Attitude Maneuvers of a Rigid Spacecraft in a Circular Orbit including Gravity Gradient Effects

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Motivation

- Attitude Dynamics of a Rigid Spacecraft in a Circular Orbit
  - Gravity gradient effects have been widely studied.
  - 24 distinct relative equilibria are exposed.
  - Linear rotational equations of motion from any relative equilibrium solutions are well known.
  - Linear attitude control including gravity gradient effects has been developed.
- Small Attitude Change Maneuvers
  - Linear analysis is applicable only to small attitude change maneuvers.
  - Local parameterization of a rotation matrix has singularities.

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- Large Attitude Change Maneuvers
  - We have to change local parameterization chart to avoid singularities.
  - Manipulation of the local chart is complicated.
  - Rotation matrix defines attitude globally without singularities.
  - Rotation matrix is convenient for algebraic manipulation.
  - Rotation matrix drifts from $\mathrm{SO}(3)$ for a numerical simulation.
- Computational Geometric Mechanics
  - Lie group variational integrator preserves the structures of $\mathrm{SO}(3)$.
  - The dynamics are linearized in Lie algebra level $\mathfrak{so}(3)$.
  - The sensitivity derivatives are used for a boundary value problem and an optimal control problem for large angle maneuvers.

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- Optimal Spacecraft Attitude Maneuver
  - Problem Formulation
  - Computational Approach
  - Numerical Example

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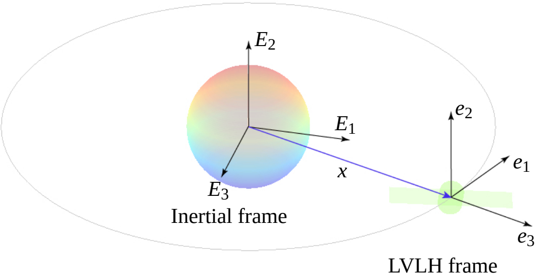
Assumption



- Basic Assumptions
  - Inertial, LVLH, and body-fixed frames are constructed.
  - Spacecraft center of mass is in circular orbit.
  - The orbital radius r, and the orbital angular velocity ω_0 is constant.
  - Spacecraft is rigid.
  - Gravity gradient moment is included.

Gravity Gradient Effects

- Non-uniform Gravitational Field
  - The gravitational field is not uniform.
  - There exist a gravitational moment about the spacecraft center of mass.
  - Significant for low earth orbit and long maneuver time cases.
- Dynamic Effects
  - 24 relative equilibria for which the principal axes are aligned with LVLH axes.
  - In a relative equilibrium, the spacecraft rotate with the orbital angular velocity.
  - The relative equilibrium is stable when $J_2 > J_1 > J_3$.
  - An orbiting spacecraft tends to align its long axis with the local vertical.

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Continuous Equations of motion

- On-orbit Spacecraft Equations of Motion

$$J\dot{\Omega} + \Omega \times J\Omega = M^g, \tag{1}$$

$$\dot{R} = RS(\Omega - \omega_0 R^T e_2), \tag{2}$$

$$M^g = 3\omega_0 R^T e_3 \times JR^T e_3, \tag{3}$$

where $S(\cdot) : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ is defined such that $S(x)y = x \times y$ for $x, y \in \mathbb{R}^3$.

- Numerical Simulation of Attitude Dynamics
  - Classical numerical integration methods consist of a linear combination of velocities.

$$R_{k+1} = R_k + h \sum_{i=1}^{s} c_i \dot{R}_{k,i}$$

  - The rotation matrix drifts from SO(3), since it is not a vector space.
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$$M_k^g = 3\omega_0 R_k^T e_3 \times J R_k^T e_3, \tag{4}$$

$$hS(J\Omega_k + \frac{h}{2} M_k^g) = F_k J_d - J_d F_k^T, \tag{5}$$

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  - The relative attitude $F_k \in \mathrm{SO}(3)$ is obtained by its Lie algebra element.
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Problem formulation

- Assumptions
  - Rest-to-rest maneuver between two given attitudes for a fixed maneuver time
  - Two impulsive control inputs are applied at the initial time and the terminal time
  - Each impulse changes the angular velocity instantaneously.
  - The motion of spacecraft between the initial time and the terminal time is uncontrolled.
- Nonlinear Boundary Value Problem

$$\text{given} : R_0, R_N^d, N$$

$$\text{find} : \Omega_0$$

$$\text{such that} : R_N = R_N^d$$

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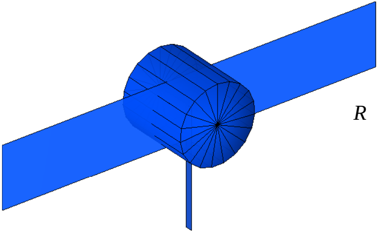
Computational Approach

- Quasi-linearization
  - The method of quasi-linearization consists of solving a sequence of linearized boundary value problems.
- Linearization
  - The linearized dynamics is small perturbations of a given trajectory.
  - The perturbation of the angular velocity; $\Omega_k^\epsilon = \Omega_k + \epsilon\delta\Omega_k$
  - The perturbation of the rotation matrix; $R_k^\epsilon \neq R_k + \epsilon\delta R_k$



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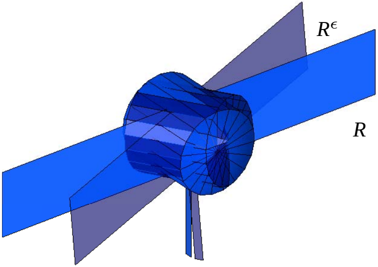

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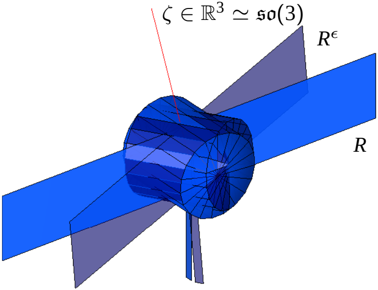
$\zeta \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$



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Computational Approach

- Linearization
  We can find linear discrete equations of motion using $\zeta_k,\ \delta\Omega_k$.

$$\begin{bmatrix}\zeta_{k+1}\\ \delta\Omega_{k+1}\end{bmatrix} = A_k \begin{bmatrix}\zeta_k\\ \delta\Omega_k\end{bmatrix},$$

  where $A_k \in \mathbb{R}^{6\times 6}$.

- Solution of a linear boundary value problem

$$\begin{bmatrix}\zeta_N\\ \delta\Omega_N\end{bmatrix} = \prod_{k=0}^{N-1} A_k \begin{bmatrix}\zeta_0\\ \delta\Omega_0\end{bmatrix},$$

$$\triangleq \begin{bmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{bmatrix} \begin{bmatrix}\zeta_0\\ \delta\Omega_0\end{bmatrix},$$

  For the given boundary value problem, $\zeta_0 = 0$ and $\delta\Omega_N$ is free. Then, we obtain

$$\zeta_N = \Phi_{12}\delta\Omega_0, \tag{12}$$

  which provides the sensitivity derivatives of the terminal attitude.

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Computational Approach

- Solution of a nonlinear boundary value problem

1. Guess an initial condition $\Omega_0^{(0)}$.
2. Set $i = 0$.
3. **while** $\|\mathrm{logm}\left(R_N^{(i)T} R_N^d\right)\| > \epsilon_S$.
4. Find $\Omega_k^{(i)}, R_k^{(i)}$ for $k = 1, 2, \cdots, N$ using a initial condition $\Omega_0^{(i)}, R_0$ (Variational Integrator Algorithm).
5. Compute the error of the boundary condition $S(\zeta_N^{(i)}) = \mathrm{logm}\left(R_N^{T(i)} R_N^d\right)$.
6. Update the initial condition, $\Omega_0^{(i+1)} = \Omega_0^{(i)} + c\Phi_{12}^{-1}\zeta_N^{(i)}$.
7. Set $i = i + 1$.
8. **end while**

where $\epsilon_S, c \in \mathbb{R}$ are a stopping criterion and a scaling factor, respectively.

Numerical Examples

- Simulation Parameters
  - Mass, length, and time dimensions are normalized.
  - Properties

$$\bar{J} = \text{diag}\left[1,\ 2.8,\ 2\right], \quad \bar{\mathrm{T}} = \frac{2\pi}{4}, \quad \epsilon_S = 10^{-14}$$

- Initial conditions
  1. Rotation maneuver about the LVLH axis e_1:

$$R_0 = I_{3\times 3}, \quad R_N^d = \text{diag}[1,\ -1,\ -1].$$

  2. Rotation maneuver about the LVLH axes e_1 and e_2:

$$R_0 = \text{diag}[1,\ -1,\ -1], \quad R_N^d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

  3. Rotation maneuver about the LVLH axes e_2 and e_3:

$$R_0 = \text{diag}[1,\ -1,\ -1], \quad R_N^d = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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(i) Rotation maneuver about the LVLH axis e_1

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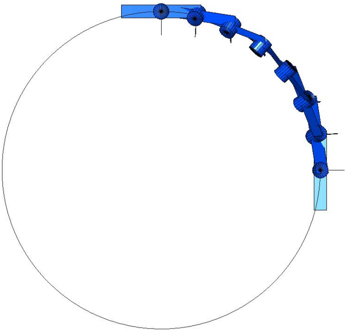
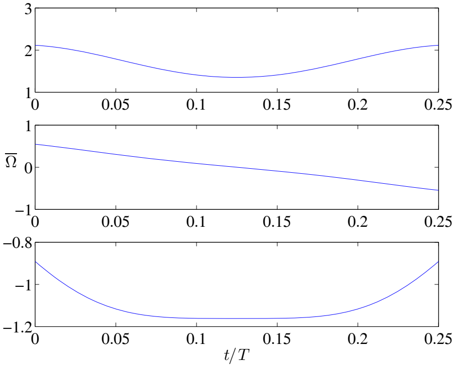


(a) Attitude Maneuver



(b) Angular velocity

Numerical Examples

(ii) Rotation maneuver about the LVLH axes e_1 and e_2

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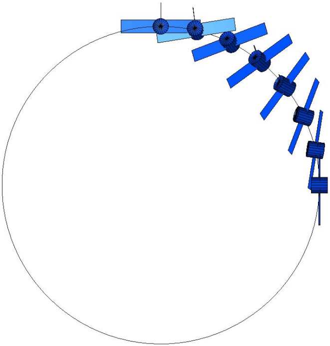

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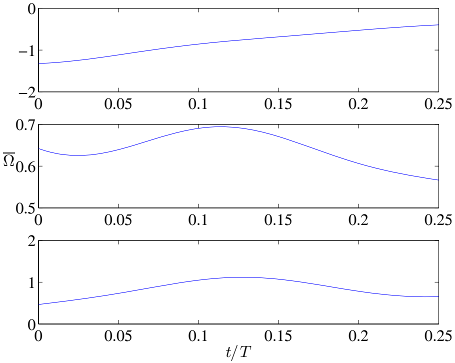

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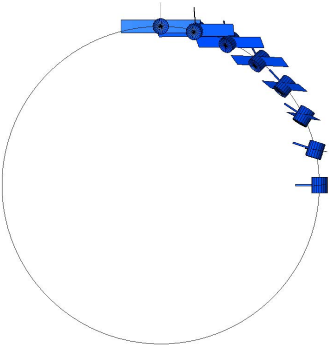
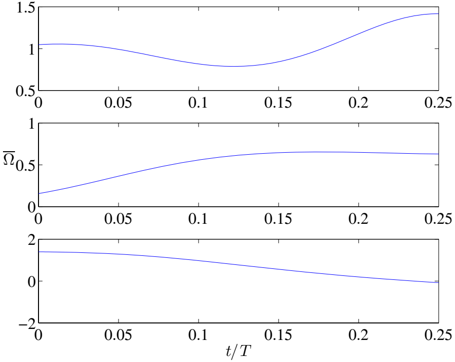
(iii) Rotation maneuver about the LVLH axes e_2 and e_3

Numerical Examples

(iii) Rotation maneuver about the LVLH axes e_2 and e_3



(a) Attitude Maneuver



(b) Angular velocity

Table of Contents

Problem formulation

- Assumptions
  - Spacecraft is axially-symmetric.
  - Reduced attitude represents the direction of the axis of symmetry. $\Lambda = Re_3 \in \mathbb{S}^2$.
  - Rest-to-rest maneuver from a given initial attitude to a terminal reduced attitude.
  - Two impulsive control inputs are applied at the initial time and the terminal time
- Optimal Attitude Maneuver Problem

$$\text{given}: R_0, \Lambda_N^d, N,$$

$$\min_{\Omega_0} \mathcal{J} = \left\| J\Omega_0 - \omega_0 R_0^T J e_2 \right\| + \left\| J\Omega_N - \omega_0 R_N^T J e_2 \right\|,$$

$$= \|H_0\| + \|H_N\|,$$

$$\text{such that}: \mathcal{C} = \left\| \Lambda_N - \Lambda_N^d \right\|^2 = 0,$$

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Computational Approach

- Sequential Quadratic Programming
  - Matlab `fmincon` function is used.
  - Analytic expressions for the gradients of the performance index $\mathcal{J}$ and the constraint $\mathcal{C}$ are used in `fmincon`.

$$\delta\mathcal{J} = \frac{H_0^T J \delta\Omega_0}{\|H_0\|} + \frac{H_N^T \left\{ J\delta\Omega_N - \omega_0 S(R_N^T J e_2)\zeta_N \right\}}{\|H_N\|}.$$

since $\delta\Omega_N = \Phi_{22}\delta\Omega_0,\ \zeta_N = \Phi_{12}\delta\Omega_0,$

$$\delta\mathcal{J} = \left[\frac{H_0^T J}{\|H_0\|} + \frac{H_N^T}{\|H_N\|} \left\{ J\Phi_{22} - \omega_0 S(R_N^T J e_2)\Phi_{12} \right\} \right] \delta\Omega_0.$$

similarly,

$$\delta\mathcal{C} = \left[2\Lambda_N^{d,T} R_N S(e_3)\Phi_{22} \right] \delta\Omega_0.$$

  - Exact computation of the gradients is crucial for efficient numerical optimization.

Numerical Examples

- Simulation Parameters
  - Mass, length, and time dimensions are normalized.
  - Properties

$$\bar{J} = \mathrm{diag}\left[3,\ 3,\ 2\right], \quad \bar{\mathrm{T}} = \frac{2\pi}{4}, \quad \epsilon_S = 10^{-14},$$
$$R_0 = \mathrm{diag}[1,\ -1,\ -1], \quad \Lambda_N^d = [0,\ -1,\ 0]^T.$$

- Optimization Results
  - Optimized performance index and violation of constraint

$$\bar{\mathcal{J}} = 6.7711,$$
$$\mathcal{C} = 4.8055 \times 10^{-14},$$
$$R_N = \begin{bmatrix} 0.6336 & 0.7736 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 \\ -0.7736 & 0.6336 & 0.0000 \end{bmatrix}.$$

Numerical Examples

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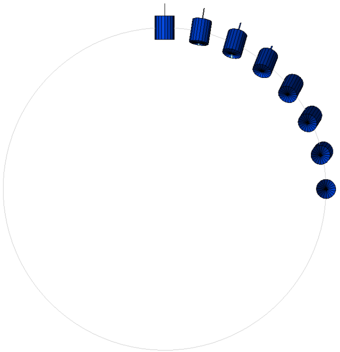

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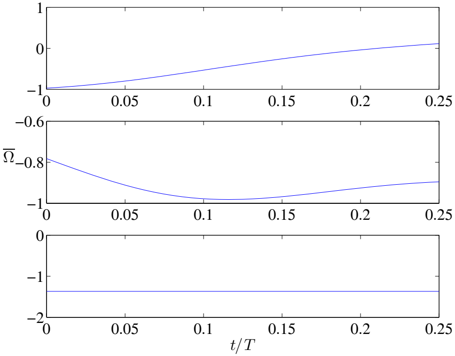

(b) Angular velocity

Sensitivity Derivatives

- Sensitivity Derivatives
  - Sensitivity derivatives are derived to be geometrically exact in the Lie algebra level.
  - Sensitivity derivatives are applied to solve a boundary value problem and an optimal attitude maneuver problem.
- Other Optimal Attitude Control Problems
  - Sensitivity derivatives w.r.t control inputs can be derived similarly.
  - Optimal control problems can be solved numerically with gradients information.
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  - Gravity gradient effects are considered.
  - Attitude is defined globally without singularities.
  - The geometric structure of the rotation matrix is preserved.
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