Spacecraft Relative Attitude Formation Tracking on $\mathsf{SO}(3)$ Based on Line-of-Sight Measurements

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Abstract—**Relative attitude formation control systems are developed for multiple spacecraft, based on the line-of-sight measurements between spacecraft in formation. The proposed control systems are unique in the sense that they do not require constructing the full attitudes of spacecraft and comparing them to obtain the relative attitudes indirectly. Instead, the control inputs are directly expressed in terms of line-of-sight measurements to control relative attitude formation precisely and efficiently. It is shown that the zero equilibrium of the relative attitude tracking errors is almost globally asymptotically stable. The desirable properties are illustrated by numerical examples, including an image processing routine to simulate vision-based line-of-sight sensors.**

I. INTRODUCTION

The coordinated control of multiple spacecraft in formation has been widely studied, as there are distinct advantages [1], [2]. Noticeable contributions on relative attitude control may be divided into leader-follower strategy [3], [4], behavior-based control [5], [6], and virtual structures [7], [8]. The aforementioned control systems for spacecraft attitude formation control have distinct features, but all of them are based on a common framework: the absolute attitude of each spacecraft with respect to an inertial frame is measured independently by using a local attitude sensor such as star trackers, and those measurements are transmitted to other spacecraft to determine relative attitudes by comparison.

This causes restrictions on the performance of coordinated spacecraft. First, *all* of spacecraft should be equipped with possibly expensive hardware systems to determine the absolute attitude completely. This may increase the overall cost of development significantly. Second, attitude formation is *indirectly* controlled by comparing the absolute attitudes of multiple spacecraft in the formation. This results in a fundamental limitation on the accuracy of attitude formation control systems, since measurement errors of multiple sensors are accumulated when determining relative attitudes.

Vision-based sensors have been widely applied for navigation of autonomous vehicles, where low-cost optical sensors are used to extract visual features to localize a vehicle [9]. In particular, it has been shown that line-of-sight (LOS) measurements between spacecraft in formation determine the relative attitudes completely. An extended Kalman filter for relative attitude is developed based on LOS observations [10]. The LOS measurements are also used for relative attitude determination of multiple vehicles [11], [12].

In this paper, a relative attitude formation control scheme is developed based on LOS measurements. Spacecraft in formation measure the LOS toward other spacecraft such that relative attitude between them asymptotically track a given desired relative attitude. Compared to other spacecraft attitude formation control systems, the proposed relative attitude control systems is unique in the sense that control inputs are directly expressed in terms of LOS measurements, and it does not require determining the full absolute attitude of spacecraft in formation or the full relative attitude between them. Therefore, relative attitudes are directly controlled, while utilizing the desirable features of vision-based sensors: they have higher accuracies at a relatively low cost, and they also have long-term stability requiring no corrections in measurements as opposed to gyros.

Compared with the preliminary work for relative attitude stabilization between two spacecraft [13], the control system proposed in this paper requires extensive analyses to take into full consideration of stability of time-varying systems for tracking, and the network structures between multiple spacecraft. The paper also provides stronger exponential stability, and numerical simulations with image processing.

Another distinct feature of the proposed relative attitude control system is that it is constructed on the special orthogonal group, $\mathsf{SO}(3)$. Attitude control systems developed on minimal representations, such as Euler-angles, have singularities, and therefore their performance for large angle rotational maneuvers is severely limited. Quaternions do not have singularities, the ambiguity in representing attitude should be carefully resolved. By following geometric control approaches [14], [15], the proposed control system is developed in a coordinate-free fashion, and it does not have any singularity or ambiguity.

II. PROBLEM FORMULATION

A. Spacecraft Attitude Formation Configuration

Consider an arbitrary number n of spacecraft in formation. Each spacecraft is considered as a rigid body, and an inertial reference frame and body-fixed frames are defined. The attitude of each spacecraft is the orientation of its body-fixed frame with respect to the inertial reference frame, and it is represented by a rotation matrix in the special orthogonal group, namely

$$\mathsf{SO}(3) = \{ R \in \mathbb{R}^{3\times3} \,|\, R^{\mathsf{T}} R = I, \quad \det R = 1 \}.$$

Each spacecraft measures the LOS from itself toward the other assigned spacecraft. A LOS observation is represented by a unit vector in the two-sphere, defined as

$$\mathsf{S}^2 = \{s \in \mathbb{R}^3 \,|\, \|s\| = 1\}.$$

For $i, j \in \{1, \ldots, n\}$ and $i \neq j$, define

$R_i \in \mathsf{SO}(3)$ the absolute attitude for the i-th space-craft, representing the linear transformation from the i-th body-fixed frame to the inertial reference frame,

$s_{ij} \in \mathsf{S}^2$ the unit vector toward the j-th spacecraft from the i-th spacecraft, represented in the inertial frame,

$b_{ij} \in \mathsf{S}^2$ the LOS direction observed from the i-th spacecraft to the j-th spacecraft, represented in the i-th body fixed frame,

$Q_{ij} \in \mathsf{SO}(3)$ the relative attitude of the i-th spacecraft with respect to the j-th spacecraft,

$Q_{ij}^d \in \mathsf{SO}(3)$ the desired relative attitude for Q_{ij}.

According to these definitions, the directions of the relative positions s_{ij} in the inertial reference frame are related to the LOS observation b_{ij} in the i-th body-fixed frame as follows:

$$s_{ij} = R_i b_{ij}, \quad b_{ij} = R_i^\mathsf{T} s_{ij}. \tag{1}$$

In short, b_{ij} represents the LOS observation of s_{ij}, observed from the i-th body. The relative attitude is given by

$$Q_{ij} = R_j^\mathsf{T} R_i, \tag{2}$$

which represents the linear transformation of the representation of a vector from the i-th body fixed frame to the j-th body-fixed frame. Note that $Q_{ij} = Q_{ji}^\mathsf{T}$.

To assign a set of LOS that should be measured for each spacecraft, a graph $(\mathcal{N}, \mathcal{E})$ is defined as follows. Each spacecraft is considered as a node, and the *set of nodes* is given by $\mathcal{N} = \{1, \ldots, n\}$. The *set of edges* $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is defined such that the relative attitude between the i-th spacecraft and the j-th spacecraft is directly controlled if $(i, j) \in \mathcal{E}$. It is undirected, i.e., $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. For each pair of two spacecraft in the edge set, another third spacecraft is assigned by the *assignment map* $\rho : \mathcal{E} \to \mathcal{N}$. As the edge set is undirected, the assignment map is symmetric, i.e., $\rho(i, j) = \rho(j, i)$.

For convenience, the edge set and the image of the assignment map are combined to form the *assignment set*:

$$\mathcal{A} = \{(i, j, k) \in \mathcal{E} \times \mathcal{N} \,|\, (i, j) \in \mathcal{E}, k = \rho(i, j)\}. \tag{3}$$

Let the *measurement set* $\mathcal{L}_i$ be the set of LOS measured from the i-th spacecraft, and let the *communication set* $\mathcal{C}_{ij}$ be the LOS transferred from the i-th spacecraft to the j-th spacecraft.

Assumption 1: The configuration of the relative positions is fixed, i.e., $\dot{s}_{ij} = 0$ for all $i, j \in \mathcal{N}$ with $i \neq j$.

Assumption 2: The third spacecraft assigned to each edge does not lie on the line joining two spacecraft connected by the edge, i.e., $s_{ik} \times s_{jk} \neq 0$ for every $(i, j, k) \in \mathcal{A}$.
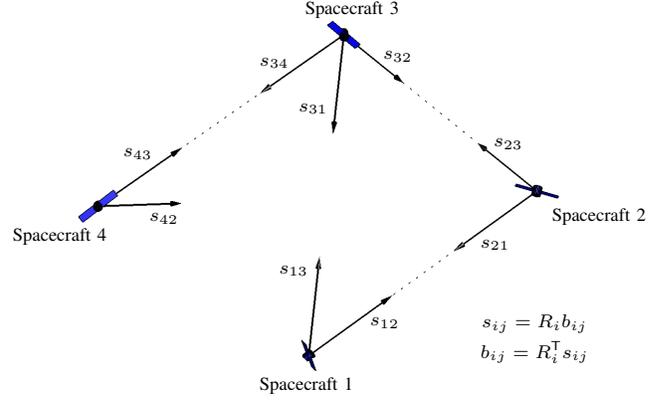

Fig. 1. Formation of four spacecraft: the direction along the relative position of the i-th body from the j-th body is denoted by s_{ij} in the inertial reference frame. The LOS observation of s_{ij} with respect the i-th body fixed frame, namely b_{ij} is obtained from (1).

Assumption 3: The measurement set of the i-th spacecraft is

$$\mathcal{L}_i = \{b_{ij}, b_{ik} \in \mathsf{S}^2 \,|\, (i, j, k) \in \mathcal{A}\}. \tag{4}$$

Assumption 4: The communication set from the i-th spacecraft to the j-th spacecraft is given by

$$\mathcal{C}_{ij} = \begin{cases} \{b_{ij}, b_{i\rho(i,j)}\} & \text{if } (i, j) \in \mathcal{E}, \\ \emptyset & \text{otherwise.} \end{cases} \tag{5}$$

Assumption 5: In the edge set, spacecraft are paired serially by daisy-chaining.

The first assumption reflects the fact that this paper does not consider the translational dynamics of spacecraft, and we focus on the rotational attitude dynamics only. The proposed control input does not depend on the values of s_{ij}, but its stability analyses is based on the first assumption that s_{ij} is fixed. The second assumption is required to determine the relative attitude between two spacecraft paired in the edge set from the assigned LOS measurements. The third assumption states that each spacecraft measures the LOS toward the paired spacecraft in the edge set, and the LOS toward the third spacecraft assigned to each pair by the assignment map. The fourth assumption implies that a spacecraft communicate only with the spacecraft paired with itself. The last assumption is made to simplify stability analysis, and the proposed relative attitude formation control system can be extended for other network topologies.

An example for formation of four spacecraft satisfying these assumptions are illustrated at Figure 1, where

$$\mathcal{A} = \{(1, 2, 3), (2, 1, 3), (2, 3, 1), (3, 2, 1), (3, 4, 2), (4, 3, 2)\}.$$

The measurement sets and the communication sets can be determined by (4) and (5) from $\mathcal{A}$. For example, for the third spacecraft, we have $\mathcal{L}_3 = \{b_{31}, b_{32}, b_{34}\}$, $\mathcal{C}_{32} = \{b_{32}, b_{31}\}$, and $\mathcal{C}_{34} = \{b_{34}, b_{32}\}$.

B. Spacecraft Attitude Dynamics

The equations of motion for the attitude dynamics of each spacecraft are given by

$$J_i \dot{\Omega}_i + \Omega_i \times J_i \Omega_i = u_i, \tag{6}$$

$$\dot{R}_i = R_i \hat{\Omega}_i, \tag{7}$$

where $J_i \in \mathbb{R}^{3 \times 3}$ is the inertia matrix of the i-th spacecraft, and $\Omega_i \in \mathbb{R}^3$ and $u_i \in \mathbb{R}^3$ are the angular velocity and the control moment of the i-th spacecraft, represented with respect to its body-fixed frame, respectively.

The *hat* map $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms a vector in $\mathbb{R}^3$ to a 3×3 skew-symmetric matrix such that $\hat{x}y = (x)^\wedge y = x \times y$ for any $x, y \in \mathbb{R}^3$. The inverse of the hat map is denoted by the *vee* map $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$. Throughout this paper, the 2-norm of a matrix A is denoted by $\|A\|$, and the dot product of two vectors is denoted by $x \cdot y = x^\mathsf{T} y$. The maximum eigenvalue and the minimum eigenvalue of J_i are denoted by λ_{M_i} and λ_{m_i}, respectively.

III. RELATIVE ATTITUDE TRACKING BETWEEN TWO SPACECRAFT

We first consider a simpler case of controlling the relative attitude between two spacecraft. Based on the results of this section, relative attitude formation control systems are developed later. As a concrete example, we develop a control system for the relative attitude between Spacecraft 1 and Spacecraft 2, namely $Q_{12} = R_2^\mathsf{T} R_1$ illustrated at Figure 1. The corresponding edge set, assignment set and measurement sets used in *this section* are given by

$$\mathcal{E} = \{(1,2), (2,1)\}, \quad \mathcal{A} = \{(1,2,3), (2,1,3)\}, \tag{8}$$
$$\mathcal{L}_1 = \mathcal{C}_{12} = \{b_{12}, b_{13}\}, \ \mathcal{L}_2 = \mathcal{C}_{21} = \{b_{21}, b_{23}\}. \tag{9}$$

Suppose that a desired relative attitude $Q_{12}^d(t)$ is given as a smooth function of time. It satisfies the kinematic equation:

$$\dot{Q}_{12}^d = Q_{12}^d \hat{\Omega}_{12}^d, \tag{10}$$

where Ω_{12}^d is the desired relative angular velocity. Note that these also yield $Q_{21}^d = (Q_{12}^d)^\mathsf{T}$ from (2), and it satisfies

$$\dot{Q}_{21}^d = Q_{21}^d \hat{\Omega}_{21}^d, \tag{11}$$

where $\Omega_{21}^d = -Q_{12}^d \Omega_{12}^d$.

The goal is to design control inputs u_1, u_2 in terms of the LOS measurements in $\mathcal{L}_1 \cup \mathcal{L}_2$ such that Q_{12} asymptotically follows Q_{12}^d, i.e., $Q_{12}(t) \to Q_{12}^d(t)$ as $t \to \infty$.

A. Kinematics of Relative Attitudes and Lines-of-Sight

For any $i, j \in \mathcal{N}$, the time-derivative of the relative attitude is given, from (7), by

$$\dot{Q}_{ij} = -\hat{\Omega}_j R_j^\mathsf{T} R_i + R_j^\mathsf{T} R_i \hat{\Omega}_i = Q_{ij}\hat{\Omega}_i - \hat{\Omega}_j Q_{ij}$$
$$= Q_{ij}(\Omega_i - Q_{ij}^\mathsf{T}\Omega_j)^\wedge \triangleq Q_{ij}\hat{\Omega}_{ij}, \tag{12}$$

where the relative angular velocity $\Omega_{ij} \in \mathbb{R}^3$ of the i-th spacecraft with respect to the j-th spacecraft is defined as

$$\Omega_{ij} = \Omega_i - Q_{ij}^\mathsf{T}\Omega_j. \tag{13}$$

From (1) and (7), the time-derivative of the LOS measurement b_{ij} is given by

$$\dot{b}_{ij} = \dot{R}_i^\mathsf{T} s_{ij} = -\hat{\Omega}_i R_i^\mathsf{T} s_{ij} = b_{ij} \times \Omega_i. \tag{14}$$

Let $b_{ijk} \in \mathbb{R}^3$ be $b_{ijk} = b_{ij} \times b_{ik}$. From (14), it can be shown that

$$\dot{b}_{ijk} = (b_{ij} \times \Omega_i) \times b_{ik} + b_{ij} \times (b_{ik} \times \Omega_i)$$
$$= -(\Omega_i \cdot b_{ik})b_{ij} + (\Omega_i \cdot b_{ij})b_{ik} = b_{ijk} \times \Omega_i. \tag{15}$$

B. Relative Attitude Tracking

It has been shown that four LOS measurements $\{b_{12}, b_{13}, b_{21}, b_{23}\}$ completely determine the relative attitude Q_{12} from the following constraints [13]:

$$b_{12} = -Q_{12}^\mathsf{T} b_{21}, \tag{16}$$
$$\frac{b_{123}}{\|b_{123}\|} = -\frac{Q_{12}^\mathsf{T} b_{213}}{\|b_{213}\|}. \tag{17}$$

These are derived from the fact that four unit vectors, namely $\{s_{12}, s_{13}, s_{21}, s_{23}\}$ lie on the sides of a triangle composed of three spacecraft. The first constraint (16) states that the unit vector from Spacecraft 1 to Spacecraft 2 is exactly opposite to the unit vector from Spacecraft 2 to Spacecraft 1, i.e., $s_{12} = -s_{21}$. The second constraint (17) implies that the plane spanned by s_{12} and s_{13} should be co-planar with the plane spanned by s_{21} and s_{23}. These geometric constraints are simply expressed with respect to the first body-fixed frame to obtain (16) and (17). For given LOS measurements $\{b_{12}, b_{13}, b_{21}, b_{23}\}$, the relative attitude Q_{12} is uniquely determined by solving (16) and (17) for Q_{12}.

We develop a relative attitude tracking control system based on these two constraints. More explicitly, control inputs are chosen such that two constraints are satisfied when the relative attitude is equal to its desired value. As both constraints are conditions on unit vectors, controller design similar to tracking control on the two-sphere. From now on, variables related to the first constraint (16) (resp., the second constraint (17)) are denoted by the sub- or super-script α (resp., β).

First, configuration error functions that represent the errors in satisfaction of (16) and (17) are defined as

$$\Psi_{12}^\alpha = \frac{1}{2}\|b_{21} + Q_{12}^d b_{12}\|^2 = 1 + b_{21} \cdot Q_{12}^d b_{12}, \tag{18}$$
$$\Psi_{12}^\beta = 1 + \frac{1}{a_{12}} b_{213} \cdot Q_{12}^d b_{123}, \tag{19}$$

where $a_{12} = a_{21} \triangleq \|b_{213}\|\|b_{123}\| \in \mathbb{R}$. Since $\|b_{ijk}\| = \|b_{ij} \times b_{ik}\| = \|R_i^\mathsf{T} s_{ij} \times R_i^\mathsf{T} s_{ik}\| = \|s_{ij} \times s_{ik}\|$, the constant a_{12} is fixed according to Assumption 1, and it is non-zero from Assumption 2. Next, we define the configuration error vectors as

$$e_{12}^\alpha = (Q_{21}^d b_{21}) \times b_{12}, \qquad e_{21}^\alpha = (Q_{12}^d b_{12}) \times b_{21}, \tag{20}$$
$$e_{12}^\beta = \frac{1}{a_{12}}(Q_{21}^d b_{213}) \times b_{123}, \quad e_{21}^\beta = \frac{1}{a_{21}}(Q_{12}^d b_{123}) \times b_{213}. \tag{21}$$

As b_{12}, b_{21} are unit vectors, and from the definition of a_{12}, a_{21}, we can show that $\|e_{12}^\alpha\|, \|e_{21}^\alpha\|, \|e_{12}^\beta\|, \|e_{21}^\beta\| \leq 1$.

We also define the angular velocity errors:

$$e_{\Omega_1} = \Omega_1 - \Omega_1^d, \quad e_{\Omega_2} = \Omega_2 - \Omega_2^d, \tag{22}$$

where the desired absolute angular velocities Ω_1^d, Ω_2^d are chosen such that

$$\Omega_{12}^d(t) = \Omega_1^d(t) - Q_{21}^d(t)\Omega_2^d(t). \tag{23}$$

Any desired absolute angular velocities satisfying (23) can be chosen. For example, they can be selected as

$$\Omega_{1_d}(t) = \frac{1}{2}\Omega_{12}^d(t), \quad \Omega_{2_d}(t) = \frac{1}{2}\Omega_{21}^d(t) = -Q_{12}^d\Omega_{1_d}(t).$$

Using these desired angular velocities, the derivative of the desired relative attitude can be rewritten as

$$\dot{Q}_{12}^d = Q_{12}^d\hat{\Omega}_1^d - \hat{\Omega}_2^d Q_{12}^d. \tag{24}$$

It is assumed that the desired angular velocities are bounded by known constants.

Assumption 6: For known positive constants B_d,

$$\|\Omega_1^d(t)\| \le B^d, \quad \|\Omega_2^d(t)\| \le B^d,$$

for all $t \ge 0$.

The properties of these error variables are summarized as follows.

Proposition 1: For positive constants $k_{12}^\alpha \ne k_{12}^\beta$, define

$$\Psi_{12} = k_{12}^\alpha \Psi_{12}^\alpha + k_{12}^\beta \Psi_{12}^\beta, \tag{25}$$
$$e_{12} = k_{12}^\alpha e_{12}^\alpha + k_{12}^\beta e_{12}^\beta, \tag{26}$$
$$e_{21} = k_{21}^\alpha e_{21}^\alpha + k_{21}^\beta e_{21}^\beta, \tag{27}$$

where $k_{21}^\alpha = k_{12}^\alpha$, $k_{21}^\beta = k_{12}^\beta$. The following properties hold:

(i) $e_{12} = -Q_{21}^d e_{21}$, and $\|e_{12}\| = \|e_{21}\|$.

(ii) $\frac{d}{dt}\Psi_{12} = e_{12} \cdot e_{\Omega_1} + e_{21} \cdot e_{\Omega_2}$.

(iii) $\|\dot{e}_{12}\| \le (k_{12}^\alpha + k_{12}^\beta)(\|e_{\Omega_1}\| + \|e_{\Omega_2}\|) + B^d\|e_{12}\|$,
$\|\dot{e}_{21}\| \le (k_{12}^\alpha + k_{12}^\beta)(\|e_{\Omega_1}\| + \|e_{\Omega_2}\|) + B^d\|e_{21}\|$.

(iv) If $\Psi_{12} \le \psi < 2\min\{k_{12}^\alpha, k_{12}^\beta\}$ for a constant ψ, then Ψ is quadratic with respect to $\|e_{12}\|$, i.e., the following inequality is satisfied:

$$\underline{\psi}_{12}\|e_{12}\|^2 \le \Psi_{12} \le \overline{\psi}_{12}\|e_{12}\|^2, \tag{28}$$

where the constants $\underline{\psi}_{12}, \overline{\psi}_{12}$ are given by

$$\underline{\psi}_{12} = \frac{\min\{k_{12}^\alpha, k_{12}^\beta\}}{2\max\{(k_{12}^\alpha)^2, (k_{12}^\beta)^2, (k_{12}^\alpha - k_{12}^\beta)^2\} + 2(k_{12}^\alpha + k_{12}^\beta)^2},$$

$$\overline{\psi}_{12} = \frac{\min\{k_{12}^\alpha, k_{12}^\beta\}(k_{12}^\alpha + k_{12}^\beta)}{\min\{(k_{12}^\alpha)^2, (k_{12}^\beta)^2\}(2\min\{k_{12}^\alpha, k_{12}^\beta\} - \psi)}.$$

Proof: See [16]. ∎

Using these properties, we develop a control system to track the given desired relative attitude as follows.

Proposition 2: Consider the attitude dynamics of spacecraft given by (6), (7) for $i \in \{1, 2\}$, with the LOS measurements specified at (8). A desired relative attitude trajectory is given by (10). For positive constants $k_{12}^\alpha \ne k_{12}^\beta, k_{21}^\alpha = k_{12}^\alpha, k_{21}^\beta = k_{12}^\beta, k_{\Omega_1}, k_{\Omega_2}$, control inputs are chosen as

$$u_i = -e_{ij} - k_{\Omega_i}e_{\Omega_i} + \hat{\Omega}_i^d J_i(e_{\Omega_i} + \Omega_i^d) + J\dot{\Omega}_i^d, \tag{29}$$

where $(i, j) \in \mathcal{E}$. Then, the following properties hold:

(i) There are four types of equilibrium, given by the desired equilibrium $(Q, \Omega_{12}) = (Q_{12}^d, \Omega_{12}^d)$, and the relative configurations represented by $Q_{12}^d = R_2^\mathsf{T} U D U^\mathsf{T} R_1$ and $\Omega_{12} = \Omega_{12}^d$ where $D \in \{\mathrm{diag}[1, -1, -1], \mathrm{diag}[-1, 1, -1], \mathrm{diag}[-1, -1, 1]\}$ and $U \in \mathsf{SO}(3)$ is the matrix composed of eigenvectors of $K_{12} = k_{12}^\alpha s_{12}s_{12}^\mathsf{T} + \frac{k_{12}^\beta}{\|\hat{s}_{12}s_{13}\|^2}(\hat{s}_{12}s_{13})(\hat{s}_{12}s_{13})^\mathsf{T}$.

(ii) The desired equilibrium is almost globally exponentially stable, and a (conservative) estimate to the region of attraction is given by

$$\Psi_{12}(0) \le \psi < 2\min\{k_{12}^\alpha, k_{12}^\beta\}, \tag{30}$$
$$\sum_{i=1,2} \lambda_{M_i}\|e_{\Omega_i}(0)\|^2 \le 2(\psi - \Psi_{12}(0)), \tag{31}$$

where ψ is a positive constant satisfying $\psi < 2\min\{k_{12}^\alpha, k_{12}^\beta\}$, and λ_{M_i} denotes the maximum eigenvalue of J_i.

(iii) The undesired equilibria are unstable.
Proof: See [16]. ∎

This states that almost all solutions of the proposed control system, excluding a class of solutions starting from a specific set that has a zero-measure, asymptotically track given the desired relative attitude. As the control inputs are expressed in terms of LOS observations, in addition to angular velocities, and the full relative attitude does not have to be constructed at each time. These results can be considered as a generalization of the preliminary work in [13], but it is a nontrivial extension as the several properties of the error variables should be considered to show a stronger exponential stability for tracking problems.

IV. RELATIVE ATTITUDE FORMATION TRACKING

The relative attitude control system between two spacecraft developed in the previous section can be used as a building block for a relative attitude formation control system for multiple spacecraft. In this section, we generalize it for daisy-chained relative attitude formation control network.

A. Relative Attitude Tracking Between Three Spacecraft

We first consider relative attitude formation tracking between three spacecraft, given by Spacecraft 1, 2, and 3, illustrated at Figure 1. The corresponding edge set and the assignment set used in this subsection are given by

$$\mathcal{E} = \{(1, 2), (2, 1), (2, 3), (3, 2)\}, \tag{32}$$
$$\mathcal{A} = \{(1, 2, 3), (2, 1, 3), (2, 3, 1), (3, 2, 1)\}. \tag{33}$$

For given relative attitude commands, $Q_{12}^d(t), Q_{23}^d(t)$, the goal is to design control inputs such that $Q_{12}(t) \to Q_{12}^d(t)$ and $Q_{23}(t) \to Q_{23}^d(t)$ as $t \to \infty$.

The definition of error variables and their properties developed in the previous section for two spacecraft are readily generalized to any $(i, j, k) \in \mathcal{A}$ in this section. For example, the kinematic equation for the desired relative attitude Q_{23}^d is obtained from (10) as

$$\dot{Q}_{23}^d = Q_{23}^d\hat{\Omega}_{23}^d,$$

where Ω_{23}^d is the desired relative angular velocity. Other configuration error functions and error vectors between Spacecraft 2 and Spacecraft 3 are defined similarly.

The desired absolute angular velocities for each spacecraft, namely Ω_1^d, Ω_2^d, and Ω_3^d should be properly defined. For the given Ω_{12}^d, Ω_{23}^d, they can be arbitrarily chosen such that

$$\Omega_{12}^d(t) = \Omega_1^d(t) - Q_{21}^d(t)\Omega_2^d(t), \tag{34}$$

$$\Omega_{23}^d(t) = \Omega_2^d(t) - Q_{32}^d(t)\Omega_3^d(t). \tag{35}$$

For example, they can be chosen as $\Omega_1^d = \Omega_{12}^d$, $\Omega_2^d = 0$, $\Omega_3^d = -Q_{23}^d\Omega_{23}^d$. Assumption 6 is considered to be satisfied such that each of the desired angular velocity is bounded by a known constant B^d.

Proposition 3: Consider the attitude dynamics of spacecraft given by (6), (7) for $i \in \{1,2,3\}$, with the LOS measurements specified at (33). Desired relative attitudes are given by $Q_{12}^d(t)$, $Q_{23}^d(t)$. For positive constants $k_{ij}^\alpha, k_{ij}^\beta, k_{\Omega_i}$ with $k_{ij}^\alpha \neq k_{ij}^\beta$, $k_{ij}^\alpha = k_{ji}^\alpha$, $k_{ij}^\beta = k_{ji}^\beta$ for $(i,j) \in \mathcal{E}$,

$$u_1 = -e_{12} - k_{\Omega_1}e_{\Omega_1} + \hat{\Omega}_1^d J_1(e_{\Omega_1} + \Omega_1^d) + J\dot{\Omega}_1^d, \tag{36}$$

$$u_2 = -\frac{1}{2}(e_{21} + e_{23}) - k_{\Omega_2}e_{\Omega_2} + \hat{\Omega}_2^d J_2(e_{\Omega_2} + \Omega_2^d) + J\dot{\Omega}_2^d, \tag{37}$$

$$u_3 = -e_{32} - k_{\Omega_3}e_{\Omega_3} + \hat{\Omega}_3^d J_3(e_{\Omega_3} + \Omega_3^d) + J\dot{\Omega}_3^d, \tag{38}$$

Then, the desired relative attitude configuration is almost globally exponentially stable, and a (conservative) estimate to the region of attraction is given by

$$\Psi_{12}(0) + \Psi_{23}(0) \leq \psi < 2\min\{k_{12}^\alpha, k_{12}^\beta, k_{23}^\alpha, k_{23}^\beta\}, \tag{39}$$

$$\lambda_{M_1}\|e_{\Omega_1}(0)\|^2 + 2\lambda_{M_2}\|e_{\Omega_2}(0)\|^2 + \lambda_{M_3}\|e_{\Omega_3}(0)\|^2 \\ \leq 2(\psi - \Psi_{12}(0) - \Psi_{23}(0)), \tag{40}$$

where ψ is a positive constant satisfying $\psi < 2\min\{k_{12}^\alpha, k_{12}^\beta, k_{23}^\alpha, k_{23}^\beta\}$.

Proof: See [16]. ∎

The control inputs for Spacecraft 1 and Spacecraft 3 at the both ends of graph are identical to (29) at Proposition 2. The control input for Spacecraft 2, which are paired with both of Spacecraft 1 and 3, is also similar to (29) except that the configuration error vectors for Spacecraft 2, namely e_{21} and e_{23} are averaged. These ideas can be generalized to relative attitude formation tracking between an arbitrary number of spacecraft as follows.

B. Relative Attitude Formation Tracking Between n Spacecraft

Consider a formation of n spacecraft, i.e., $\mathcal{N} = \{1, \ldots, n\}$. According to Assumption 5, spacecraft are paired serially in the edge set. For convenience, it is assumed that spacecraft are numbered such that the edge set is given by

$$\mathcal{E} = \{(1,2), \ldots, (n-1,n), (2,1), \ldots (n,n-1)\}. \tag{41}$$

The assignment set is given by (3) for an arbitrary assignment map satisfying Assumption 2. The desired relative attitudes Q_{ij}^d for $(i,j) \in \mathcal{E}$ are prescribed. The definition of error variables and their properties developed in Section III are
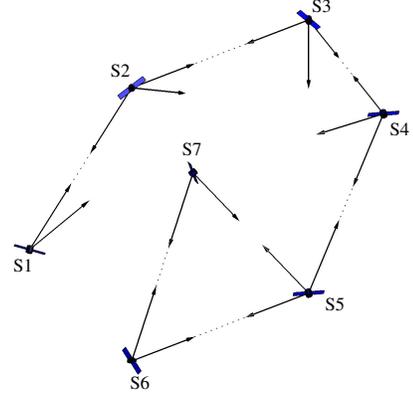

Fig. 2. Relative attitude formation tracking for 7 spacecraft: the lines-of-sight measured by each spacecraft are denoted by arrows, and the dotted line between two spacecraft implies that they are paired at the edge set.

generalized to any $(i,j,k) \in \mathcal{A}$. The desired absolute angular velocities Ω_i^d for $i \in \mathcal{N}$ are chosen such that

$$\Omega_{ij}^d(t) = \Omega_i^d(t) - Q_{ji}^d(t)\Omega_j^d(t) \quad \text{for } (i,j) \in \mathcal{E}. \tag{42}$$

Proposition 4: Consider the attitude dynamics of spacecraft given by (6), (7) for $i \in \{1, \ldots, n\}$, with the LOS measurements specified by (41),(3). Desired relative attitudes are given by $Q_{ij}^d(t)$ for $(i,j) \in \mathcal{E}$. For positive constants $k_{ij}^\alpha, k_{ij}^\beta, k_{\Omega_i}$ with $k_{ij}^\alpha \neq k_{ij}^\beta$, $k_{ij}^\alpha = k_{ji}^\alpha$, $k_{ij}^\beta = k_{ji}^\beta$, the control inputs chosen as

$$u_1 = -e_{12} - k_{\Omega_1}e_{\Omega_1} + \hat{\Omega}_1^d J_1(e_{\Omega_1} + \Omega_1^d) + J\dot{\Omega}_1^d, \tag{43}$$

$$u_p = -\frac{1}{2}(e_{p,p-1} + e_{p,p+1}) - k_{\Omega_p}e_{\Omega_p} + \hat{\Omega}_p^d J_p(e_{\Omega_p} + \Omega_p^d) \\ + J\dot{\Omega}_p^d \quad \text{for } p \in \{2, \ldots, n-1\}, \tag{44}$$

$$u_n = -e_{n,n-1} - k_{\Omega_n}e_{\Omega_n} + \hat{\Omega}_n^d J_n(e_{\Omega_n} + \Omega_n^d) + J\dot{\Omega}_n^d. \tag{45}$$

Then, the desired relative attitude configuration is almost globally exponentially stable.

Proof: See [16]. ∎

V. NUMERICAL EXAMPLE

Consider the formation of seven spacecraft illustrated at Figure 2. The corresponding edge is given by (41) with $n = 7$, and the assignment set is

$$\mathcal{A} = \{(1,2,3), (2,1,3), (2,3,4), (3,2,4), (3,4,5), (4,3,5),$$
$$(4,5,7), (5,4,7), (5,6,7), (6,5,7), (6,7,5), (7,6,5)\}.$$

The desired relative attitudes for Q_{34}^d and Q_{45}^d are given in terms of 3-2-1 Euler angles as $Q_{34}^d(t) = Q_{34}^d(\alpha(t), \beta(t), \gamma(t))$, $Q_{45}^d(t) = Q_{45}^d(\phi(t), \theta(t), \psi(t))$, where

$$\alpha(t) = \sin 0.5t, \ \beta(t) = 0.1, \ \gamma(t) = \cos t,$$

$$\phi(t) = 0, \ \theta(t) = -0.1 + \cos 0.2t, \ \psi(t) = 0.5\sin 2t,$$

and $Q_{12}^d(t) = Q_{23}^d(t) = Q_{56}^d(t) = I$, $Q_{67}^d(t) = (Q_{45}^d(t))^{\mathsf{T}}$. It is chosen that $\Omega_4^d(t) = 0$, and other desired absolute angular velocities are selected to satisfy (42).

The initial attitudes for Spacecraft 3 and 6 are chosen as $R_3(0) = \exp(0.999\pi\hat{e}_1)$ and $R_6(0) = \exp(0.990\pi\hat{e}_2)$,
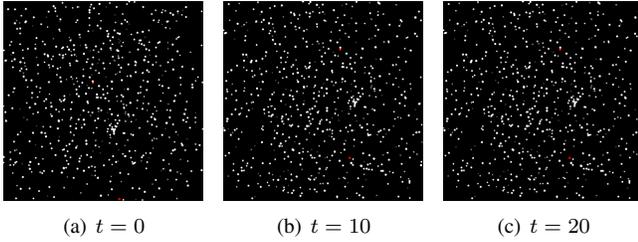
(a) $t = 0$ (b) $t = 10$ (c) $t = 20$

Fig. 3. Virtual image observed from Spacecraft 4: spacecraft and distant stars are denoted by red dots and white dots, respectively.

where $e_1 = [1, 0, 0]^\mathsf{T}, e_2 = [0, 1, 0]^\mathsf{T} \in \mathbb{R}^3$. The initial attitudes for other spacecraft are chosen as the identity matrix. The resulting initial errors for the relative attitudes Q_{23} and Q_{67} are $0.99\pi \, \mathrm{rad} = 179.82°$. The initial angular velocity is chosen as zero for every spacecraft.

The inertia matrix is identical, i.e., $J_i = \mathrm{diag}[3, 2, 1] \, \mathrm{kgm}^2$ for all $i \in \mathcal{N}$. Controller gains are chosen as $k_{\Omega_i} = 7$, $k_{ij}^\alpha = 25$, and $k_{ij}^\beta = 25.1$ for any $(i, j) \in \mathcal{E}$.

In numerical simulations, an image processing routine is incorporated. A virtual image observed from Spacecraft 4 is generated based on the relative positions and its attitude R_4. The generated image is processed online to find the locations of Spacecraft 3 and 7 in the two-dimensional image plane, and it is transformed into the body-fixed frame of Spacecraft 4 to obtain b_{43}, b_{47}. This is to test the feasibility of the proposed vision-based spacecraft relative attitude formation control scheme with numerical simulations integrated with image processing.

The corresponding numerical results are illustrated at Figures 4 and 3. Tracking errors for relative attitudes and control inputs are shown at Figure 4, where the relative attitude error vectors are defined as $e_{Q_{ij}} = \frac{1}{2}((Q_{ij}^d)^\mathsf{T} Q_{ij} - Q_{ij}^\mathsf{T} Q_{ij}^d)^\vee \in \mathbb{R}^3$. These illustrate good convergence rates. Virtual images observed from Spacecraft 4 at few times are also shown at Figure 3.

(a) Relative attitude error functions $\Psi_{12}, \Psi_{23}, \ldots, \Psi_{67}$

(b) Relative attitude error vectors $e_{Q_{12}}, e_{Q_{23}}, \ldots, e_{Q_{67}}$

(c) Relative angular velocity error $e_{\Omega_1}, \ldots, e_{\Omega_7}$

(d) Control moments $u_1, \ldots, u_7$

Fig. 4. Numerical results for seven spacecraft in formation (blue, green, red, cyan, magenta, and black in ascending order)

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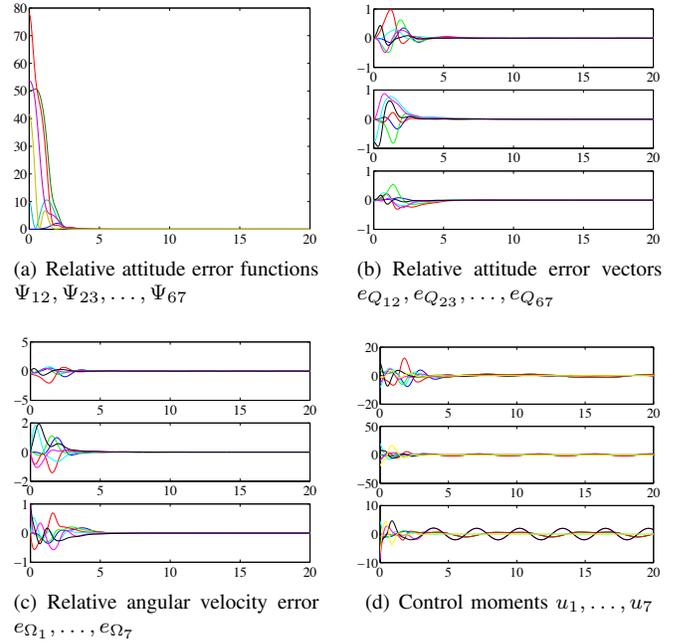
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