Relative Attitude Control of Two Spacecraft on $\mathsf{SO}(3)$ using Line-of-Sight Observations

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Abstract— **This paper studies relative attitude control of two spacecraft, considered as rigid bodies. A relative attitude determination scheme and a relative attitude control system are developed based on the line-of-sight (LOS) observations between two spacecraft, and another set of LOS observations to a common object from two spacecraft. The proposed control system is distinct from other attitude formation control systems that require the full absolute attitude of spacecraft and the full relative attitude between them. Assuming that the relative positions of two spacecraft and the common object are fixed, it is shown that the desired relative attitude between two spacecraft is almost globally asymptotically stable. A numerical example is presented to illustrate the desirable properties of the proposed relative attitude control system.**

I. Introduction

Relative attitude control problems have been investigated with applications to spacecraft formation flight [1]. Nonlinear attitude tracking control for spacecraft formation is studied in [2]. Attitude tracking and formation controls without angular velocity measurements are addressed in [3]. An adaptive formation controller is studied in [4]. A behavior-based control approach is considered in [5].

However, these approaches assume that the full attitude of each spacecraft is available, and they are transmitted to other spacecraft to obtain the relative attitudes required for a formation control system. For example, the global positioning system (GPS) has been used to determine relative attitudes and positions, along with inertial measurement units [6]. But, this requires extensive hardware developments, and any GPS signal is subject to several limitations such as interference and jamming.

Therefore, the development of relative configuration estimators, that is independent of external systems such as GPS, is investigated [7]. The current research includes a vision-based navigation system that is composed of optical sensors [8]. This has desirable properties that its hardware structure is relatively simple, and it has low-costs. Recently, an extended Kalman filter for relative spacecraft attitude is developed based on line-of-sight (LOS) observations of a vision-based navigation system [9]. LOS measurements are also used for relative attitude determination problems for multiple vehicle configurations [10], [11].

This paper is motivated by the relative attitude determination scheme studied in [11], where the relative attitude of two vehicles are determined by LOS observations between them and another set of LOS observations to a common object measured by both vehicles. Here, we consider a relative attitude control problem based on LOS observations. The relative attitude of two spacecraft is asymptotically stabilized to a desired relative attitude, and the control input is expressed directly in terms of four LOS observations, namely two LOS observations between two spacecraft, and two LOS observations toward a common object. Assuming that the relative locations of two spacecraft and the common object are fixed, it is shown that the exact location of any of them is not required.

Compared to other spacecraft attitude formation control systems, the proposed relative attitude control systems is designed based on vision-based LOS observations, and it does not require determining the full absolute attitude of both spacecraft or the full relative attitude between them.

Another distinct feature of the proposed relative attitude control system is that it is constructed on the special orthogonal group, $\mathsf{SO}(3)$. Attitude control systems developed on minimal representations, such as Euler-angles, have singularities, and therefore their performance for large angle rotational maneuvers is severely limited. Quaternions do not have singularities, but as the three-sphere double-covers the special orthogonal group, the ambiguity in representing attitude should be carefully resolved in any quaternion-based attitude control system. By following geometric control approaches [12], [13], the proposed control system is developed in a coordinate-free fashion, and it does not have any singularity or ambiguity.

This paper is organized as follows. Spacecraft configurations are defined in Section II. A relative attitude determination scheme using LOS observations is presented in Section III, and a relative attitude control system is developed in Section IV, followed by a numerical example.

II. Problem Formulation

A. Spacecraft Configuration

Consider two spacecraft and an arbitrary object, as illustrated at Figure 1. We consider each spacecraft is a rigid body, and we define an inertial reference frame and two body-fixed frames. The attitude of each spacecraft is the orientation of its body-fixed frame with respect to the inertial reference frame, and it is represented by a rotation matrix in the special orthogonal group, namely

$$\mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times 3}\,|\, R^T R = I,\quad \det R = 1\}.$$

Each spacecraft observes the LOS from itself to the other spacecraft, and it also observes the LOS to a common object,

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*This research has been supported in part by NSF under the grant CMMI-1029551.

such as a distant star. A LOS observation is represented by a unit vector in the two-sphere, defined as

$$\mathsf{S}^2 = \{s \in \mathbb{R}^3 \,|\, \|s\| = 1\}.$$

Two spacecraft are denoted by the subscripts 1 and 2, and the common object is denoted by the subscript 3. Define

$R_i \in \mathsf{SO(3)}$	the rotation matrix for the i-th spacecraft, representing the linear transformation from the i-th body-fixed frame to the inertial reference frame ($1 \leq i \leq 2$)
$s_{ij} \in \mathsf{S}^2$	the direction of the relative position of the j-th body from the i-th body, represented in the inertial frame ($(i,j) \in \{(1,2),(1,3),(2,1),(2,3)\}$)
$b_{ij} \in \mathsf{S}^2$	the LOS direction observed from the i-th body to the j-th body, represented in the i-th body fixed frame ($(i,j) \in \{(1,2),(1,3),(2,1),(2,3)\}$)
$Q \in \mathsf{SO(3)}$	the relative attitude of Spacecraft 1 with respect to Spacecraft 2, representing the linear transformation from the first body-fixed frame to the second body-fixed frame, i.e $Q = R_2^T R_1$.
$Q_d \in \mathsf{SO(3)}$	the desired relative attitude

According to these definitions, the directions of the relative positions s_{ij} in the inertial reference frame are related to the LOS observation b_{ij} in the i-th body-fixed frame as follows:

$$s_{ij} = R_i b_{ij}, \quad b_{ij} = R_i^T s_{ij} \tag{1}$$

for $(i,j) \in \{(1,2),(1,3),(2,1),(2,3)\}$. In short, b_{ij} represents the LOS observation of s_{ij}, observed from the i-th body.

The results of this paper are based on the following assumptions.

Assumption 1: The relative positions between two spacecraft and the common object are fixed, i.e. $\dot{s}_{ij} = 0$ for $(i,j) \in \{(1,2),(1,3),(2,1),(2,3)\}$.

Assumption 2: The common object does not lie on the line joining Spacecraft 1 and Spacecraft 2, i.e. $s_{12} \times s_{13} \neq 0$ or $s_{21} \times s_{23} \neq 0$.

Assumption 3: Four LOS observations, namely $\{b_{12},\, b_{13},\, b_{21},\, b_{23}\}$, are available for both spacecraft.

The first assumption reflects the fact that this paper does not consider the translational dynamics of spacecraft, and we focus on the rotational attitude dynamics only. The second assumption is required when determining the relative attitude based on the four LOS observations. There is no further requirement about the third common object. The choice of the common object is completely arbitrary, and any distinctive object can be chosen as long as it does not violate the second assumption. Its position is possibly unknown, and there is no need to communicate with the common object. The last assumption is required as the control input at each spacecraft is expressed in terms of those LOS observations. Therefore, two LOS observations of each spacecraft should be transmitted to the other spacecraft.

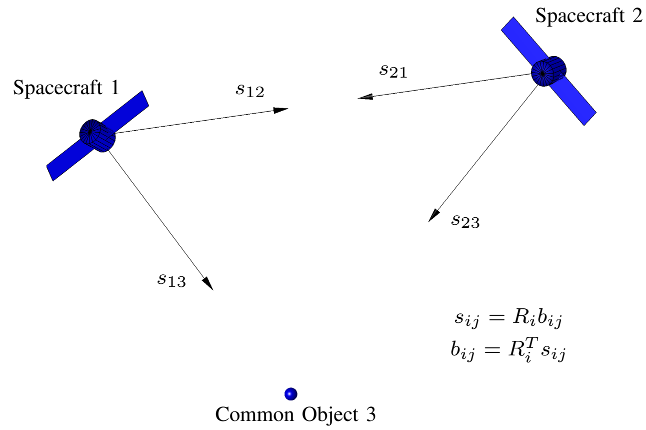


Fig. 1. Formation of two spacecraft: the direction along the relative position of the i-th body from the j-th body is denoted by s_{ij} in the inertial reference frame. The LOS observation of s_{ij} with respect the i-th body fixed frame, namely b_{ij} is obtained from (1).

B. *Spacecraft Attitude Dynamics*

The equations of motion for the attitude dynamics of each spacecraft are given by

$$J_i\dot{\Omega}_i + \Omega_i \times J_i\Omega_i = u_i, \tag{2}$$

$$\dot{R}_i = R_i\hat{\Omega}_i, \tag{3}$$

where $J_i \in \mathbb{R}^{3\times3}$ is the inertia matrix of the i-th spacecraft, and $\Omega_i \in \mathbb{R}^3$ and $u_i \in \mathbb{R}^3$ are the angular velocity and the control moment of the i-th spacecraft, represented with respect to its body fixed frame, respectively.

The *hat* map $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms a vector in $\mathbb{R}^3$ to a 3×3 skew-symmetric matrix such that $\hat{x}y = x \times y$ for any $x, y \in \mathbb{R}^3$. The inverse of the hat map is denoted by the *vee* map $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$. Few properties of the hat map are summarized as follows:

$$\hat{x}y = x \times y = -y \times x = -\hat{y}x, \tag{4}$$

$$R\hat{x}R^T = (Rx)^\wedge, \tag{5}$$

$$x \cdot \hat{y}z = y \cdot \hat{z}x = z \cdot \hat{x}y, \tag{6}$$

for any $x, y, z \in \mathbb{R}^3$, and $R \in \mathsf{SO(3)}$. Throughout this paper, the 2-norm of a matrix A is denoted by $\|A\|$, and the dot product of two vectors is denoted by $x \cdot y = x^T y$.

III. Relative Attitude Determination

In this section, we consider an algorithm to determine the relative attitude $Q \in \mathsf{SO(3)}$ between two spacecraft for given four LOS observations, namely $b_{12}, b_{13}, b_{21}, b_{23}$. In particular, we present two constraints that describe the relationship between the relative attitude and the LOS observations.

1) Constraint for the LOS observations between spacecraft: According to the given configuration of spacecraft, illustrated at Figure 1, it is straightforward to show

$$s_{12} = -s_{21},$$

which states that the direction from Spacecraft 1 to Spacecraft 2 is exactly opposite to the direction from Spacecraft 2 to Spacecraft 1.

This can be rewritten in terms of the LOS observations as follows. Multiplying both sides by $R_1^T = Q^T R_2^T$, we obtain

$$R_1^T s_{12} = -Q^T R_2^T s_{21}.$$

From (1), this is written as

$$b_{12} = -Q^T b_{21}, \quad \text{(or equivalently } Qb_{12} = -b_{21}). \tag{7}$$

For given LOS observations between spacecraft, namely b_{12}, b_{21}, equation (7) provides a two-dimensional constraint on the three-dimensional relative attitude Q. Any rotation matrix Q satisfying (7) has a one-dimensional degree of freedom, which corresponds to the rotation of each spacecraft about s_{12} or s_{21}. This is resolved by the next constraint.

2) Constraint for the LOS observation to the common object: This constraint is based on the fact that all of the relative position vectors should lie in the plane defined by two spacecraft and the common object. In [11], a constraint on the LOS observations is derived using the fact that the summation of the inner angles of the triangle is equal to π. In this paper, we introduce another form of the constraint to resolve the ambiguity of (7). The proposed constraint is based on the direction of two vectors, and it is more straightforward to develop a relative attitude control scheme based the proposed, direction based constraint.

According to Figure 1, the plane spanned by s_{12} and s_{13} is coplanar with the plane spanned by s_{21} and s_{23}:

$$\frac{s_{12} \times s_{13}}{\|s_{12} \times s_{13}\|} = -\frac{s_{21} \times s_{23}}{\|s_{21} \times s_{23}\|}.$$

From Assumption 2, this expression is well-defined. Multiplying both sides by $R_1^T = Q^T R_2^T$, and using (1),

$$\frac{b_{12} \times b_{13}}{\|b_{12} \times b_{13}\|} = -\frac{Q^T(b_{21} \times b_{23})}{\|b_{21} \times b_{23}\|}, \tag{8}$$

which is another constraint that describes the relationship between the relative attitude and the LOS observations.

The two constraints (7), (8) can be interpreted as follows. Let $\mathcal{P}_1$, $\mathcal{P}_2$ denote two planes defined as

$$\mathcal{P}_1 : \text{plane spanned by } b_{12} \text{ and } b_{13}, \tag{9}$$

$$\mathcal{P}_2 : \text{plane spanned by } Q^T b_{21} \text{ and } Q^T b_{23}. \tag{10}$$

Note that these planes $\mathcal{P}_1$, $\mathcal{P}_2$ are defined with respect to the body-fixed frame of Spacecraft 1. The second constraint (8) is to guarantee that $\mathcal{P}_1$ and $\mathcal{P}_2$ are coplanar, and the first constraint (7) is to guarantee that the vector b_{12} on $\mathcal{P}_1$ is exactly opposite to the vector $Q^T b_{21}$ on $\mathcal{P}_2$.

Each constraint is two-dimensional, but they altogether provide a three-dimensional constraint to find the unique relative attitude Q satisfying both constraints. For example, if the two-dimensional constraint (7) is satisfied, then the intersection between $\mathcal{P}_1$ and $\mathcal{P}_2$ is along the direction of $b_{12} = -Q^T b_{21}$, and (8) provides an additional one-dimensional constraint to make $\mathcal{P}_1$ and $\mathcal{P}_2$ coplanar by rotating them about $b_{12} = -Q^T b_{21}$.

If (8) is satisfied, then $\mathcal{P}_1$, $\mathcal{P}_2$ are guaranteed to be coplanar, and (7) provides an additional one-dimensional constraint to align b_{12} with $-Q^T b_{21}$ within the plane $\mathcal{P}_1 = \mathcal{P}_2$. Therefore, for a given set of four LOS observations, we can find the unique relative attitude Q satisfying (7) and (8). An explicit expression for such Q is presented in [11]. In this paper, we mainly use these two constraints to develop a relative attitude control system in the next section.

IV. Relative Attitude Control

Suppose that a desired relative attitude between two spacecraft, namely $Q_d \in \mathsf{SO}(3)$, is given. In this section, we find an expression for the control inputs, u_1, u_2, such that the relative attitude Q is asymptotically stabilized to the desired attitude Q_d, i.e. $Q(t) \to Q_d$ as $t \to \infty$. The key idea is that we choose the control inputs such that two constraints, (7) and (8) are satisfied when the relative attitude is equal to the desired relative attitude.

In this paper, the subscript α denotes variables related to the first constraint (7), and the subscript β denotes variables relevant to the second constraint (8).

A. Relative Attitude Error Functions

Both constraints are conditions on matching the directions of two unit vectors. For example, to satisfy the first constraint (7) at the desired relative attitude, the direction of the unit vector b_{12} should be the same as the unit vector $-Q_d^T b_{21}$. Similarly, the second constraint (8) is also a condition on the direction of two unit vectors. We choose control inputs such that those two pairs of the unit vectors are matched with each other, and the resulting control input design procedure is similar to stabilization on the two-sphere [13], [14].

1) Error function for the LOS observations between spacecraft: For the first constraint (7), we define the following error function:

$$\Psi_\alpha = \frac{1}{2}\|b_{21} + Q_d b_{12}\|^2 = 1 + b_{21} \cdot Q_d b_{12}. \tag{11}$$

This represents $\Psi_\alpha = 1 - \cos\theta_\alpha$, where θ_α is the angle between b_{21} and $-Q_d b_{12}$, and it is positive definite about $b_{21} = -Q_d b_{12}$, and it is uniformly quadratic [13]. We have $0 \leq \Psi_\alpha \leq 2$, and the critical points of Ψ_α correspond to the configurations when $\Psi_\alpha = 0$ or $\Psi_\alpha = 2$.

According to Assumption 1, Ψ_α can be considered as a function of R_1 and R_2 as follows. Substituting (1), we obtain

$$\Psi_\alpha(R_1, R_2) = 1 + s_{21} \cdot R_2 Q_d R_1^T s_{12}.$$

Since the unit vectors, s_{12}, s_{21} are fixed, the error function Ψ_α depends on the attitude of spacecraft, R_1, R_2.

To choose the control inputs, we find the derivative of Ψ_α with respect to R_1, R_2. The infinitesimal variation of a rotation matrix can be expressed in terms of the exponential map as follows:

$$\delta R = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} R\exp(\epsilon\hat{\eta}) = R\hat{\eta},$$

for a vector $\eta \in \mathbb{R}^3$. The variation of Ψ_α is given by

$$\begin{aligned}\delta\Psi_\alpha &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Psi_\alpha(R_1\exp(\epsilon\hat{\eta}_1), R_2\exp(\epsilon\hat{\eta}_2)) \\ &= -s_{21} \cdot R_2 Q_d \hat{\eta}_1 R_1^T s_{12} + s_{21} \cdot R_2 \hat{\eta}_2 Q_d R_1^T s_{12}\end{aligned}$$

$$= -b_{21} \cdot Q_d \hat{\eta}_1 b_{12} + b_{21} \cdot \hat{\eta}_2 Q_d b_{12}, \tag{12}$$

where (1) is used at the last step. From (6), we obtain

$$\begin{aligned} \delta\Psi_\alpha &= (Q_d^T b_{21} \times b_{12}) \cdot \eta_1 + (Q_d b_{12} \times b_{21}) \cdot \eta_2, \\ &\triangleq \psi_{\alpha_1} \cdot \eta_1 + \psi_{\alpha_2} \cdot \eta_2, \end{aligned} \tag{13}$$

where $\psi_{\alpha_1}, \psi_{\alpha_2} \in \mathbb{R}^3$.

2) Error Function for the LOS observation to the common object: Similarly, we define another error function for (8):

$$\begin{aligned} \Psi_\beta(R_1, R_2) &= 1 + \frac{1}{c} Q_d (b_{12} \times b_{13}) \cdot (b_{21} \times b_{23}) \qquad (14) \\ &= 1 + \frac{1}{c} Q_d R_1^T (s_{12} \times s_{13}) \cdot R_2^T (s_{21} \times s_{23}), \end{aligned}$$

where the constant c is given by

$$c = \|b_{12} \times b_{13}\| \|b_{21} \times b_{23}\| = \|s_{12} \times s_{13}\| \|s_{21} \times s_{23}\|. \tag{15}$$

The second equality of (15) is from (1), (5). According to Assumption 1, it is guaranteed that c is fixed, and from Assumption 2, we have $c \neq 0$. This error function Ψ_β represents $\Psi_\beta = 1 - \cos\theta_\beta$, where θ_β is the angle between $Q_d(b_{12} \times b_{13})$ and $-b_{21} \times b_{23}$, and it is positive definite about the relative attitude Q satisfying (8).

By following the same approach presented at (12), (13), the variation of Ψ_β can be written as

$$\delta\Psi_\beta = \psi_{\beta 1} \cdot \eta_1 + \psi_{\beta 2} \cdot \eta_2, \tag{16}$$

where $\psi_{\beta_1}, \psi_{\beta_2} \in \mathbb{R}^3$ are given by

$$\psi_{\beta_1} = \frac{1}{c} (Q_d^T (b_{21} \times b_{23})) \times (b_{12} \times b_{13}), \tag{17}$$

$$\psi_{\beta_2} = \frac{1}{c} (Q_d (b_{12} \times b_{13})) \times (b_{21} \times b_{23}). \tag{18}$$

B. Relative Attitude Control System

For positive constants $k_{\Omega_1}, k_{\Omega_2}, k_\alpha, k_\beta$, we choose the control inputs as follows:

$$\begin{aligned} u_1 = &-k_{\Omega_1} \Omega_1 - k_\alpha (Q_d^T b_{21} \times b_{12}) \\ &- k_\beta \frac{1}{c} (Q_d^T (b_{21} \times b_{23})) \times (b_{12} \times b_{13}), \end{aligned} \tag{19}$$

$$\begin{aligned} u_2 = &-k_{\Omega_2} \Omega_2 - k_\alpha (Q_d b_{12} \times b_{21}) \\ &- k_\beta \frac{1}{c} (Q_d (b_{12} \times b_{13})) \times (b_{21} \times b_{23}). \end{aligned} \tag{20}$$

The first terms correspond to dissipation, and the remaining two terms of each control input are chosen to minimize the error functions, Ψ_α and Ψ_β, respectively, and they are directly expressed in terms of $\psi_{\alpha_1}, \psi_{\alpha_2}, \psi_{\beta_1}, \psi_{\beta_2}$. The corresponding stability properties are summarized as follows.

Proposition 4: Consider the attitude dynamics of two spacecraft given by (2), (3), and they satisfy Assumptions 1-3. For a given desired attitude $Q_d \in \mathsf{SO}(3)$, the control inputs are chosen as (19), (20) with positive constants $k_{\Omega_1}, k_{\Omega_2}, k_\alpha, k_\beta$ satisfying $k_\alpha \neq k_\beta$. Then, the following properties hold:

(i) there are four equilibrium configurations defined by

$$E = \{(R_1, R_2) \in \mathsf{SO}(3)^2 \,|\, b_{12} = \pm Q_d^T b_{21},$$

$$\frac{b_{12} \times b_{13}}{\|b_{12} \times b_{13}\|} = \pm \frac{Q_d^T (b_{21} \times b_{23})}{\|b_{21} \times b_{23}\|}\}. \tag{21}$$

(ii) the desired equilibrium $(Q, \Omega_1, \Omega_2) = (Q_d, 0, 0)$ is almost globally asymptotically stable, where an estimate to the region of attraction is given by

$$k_\alpha \Psi_\alpha(0) + k_\beta \Psi_\beta(0) \leq \psi < \min\{2k_\alpha,\, 2k_\beta\}, \tag{22}$$

$$\begin{aligned} &\sum_{i=1}^{2} \lambda_{\max}(J_i) \|\Omega_i(0)\|^2 \\ &\quad < \min\{2k_\alpha,\, 2k_\beta\} - (k_\alpha \Psi_\alpha(0) + k_\beta \Psi_\beta(0)). \end{aligned} \tag{23}$$

(iii) three undesired equilibria are unstable.

Proof: See Appendix. ■

This states that almost all solutions of the proposed control system, excluding a class of solutions starting from a specific set that has a zero-measure, asymptotically converge to the desired relative attitude. At (19), (20), the control inputs are expressed in terms of four LOS observations, in addition to angular velocities, and the full relative attitude does not have to be constructed at each time. The control inputs also do not directly use the relative positions between two spacecraft and the common object at all. Furthermore, as discussed in Section II, the choice of the common object is arbitrary, and there is no need for communication between spacecraft and the common object.

V. Numerical Examples

Consider two spacecraft with the identical moment of inertia matrix, given by $J_1 = J_2 = \mathrm{diag}[0.23,\, 0.28,\, 0.35]\,\mathrm{kgm}^2$. The configuration of two spacecraft and the common object is chose such that

$$\begin{aligned} s_{12} &= [1,\, 0,\, 0]^T = -s_{21}, \\ s_{13} &= [0.1761,\, -0.8805,\, 0.4402]^T, \\ s_{23} &= [-0.5819,\, -0.7274,\, 0.3637]^T. \end{aligned}$$

The desired relative attitude is given by $Q_d = I$, so this example is for attitude synchronization. We consider a worst initial relative attitude configuration given by

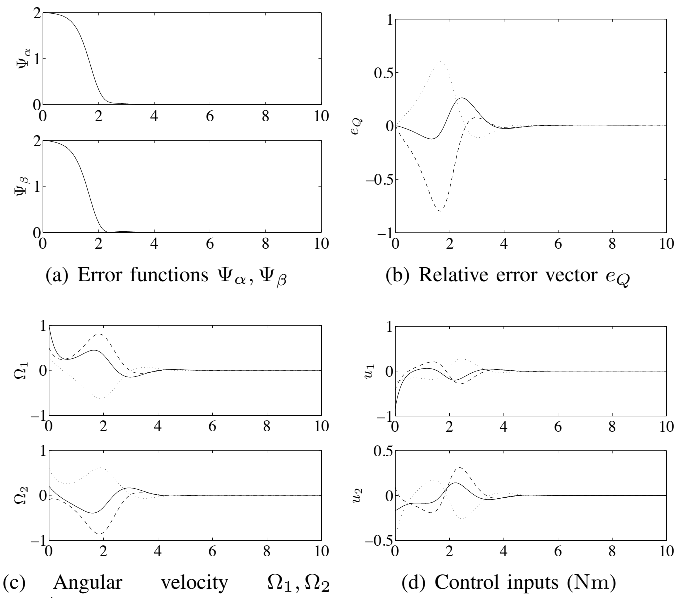
$$R_1(0) = \exp(\pi \hat{s}_{12}), \quad R_2(0) = \exp(0.99\pi \hat{s}_{123}),$$

where $s_{123} = (s_{12} \times s_{23})/\|s_{12} \times s_{13}\| \in \mathsf{S}^2$. The corresponding initial values of the relative attitude error functions are given by $\Psi_\alpha(0) = 1.9995$, $\Psi_\beta(0) = 2$. The initial angular velocities are chosen as $\Omega_1(0) = [1,\, 0.5,\, 0.3]^T$, $\Omega_2(0) = [0.2,\, -0.1,\, 0.6]^T$ (rad/s). The control inputs parameters are chosen as $k_{\Omega_1} = k_{\Omega_2} = 0.84$, $k_\alpha = 0.5250$, $k_\beta = 0.5775$.

Simulation results are illustrated at Figure 2. At Figures 2(a),(b), the relative attitude error functions, Ψ_α, Ψ_β, and the following additional relative attitude error vector are shown: $e_Q = \frac{1}{2}(Q_d^T Q - Q^T Q_d)^\vee$. These illustrate the convergence of the relative attitude to the desired value. The actual attitudes of two spacecraft at 10 seconds are given by

$$R_1(10) = \begin{bmatrix} 0.0547 & 0.6200 & 0.7827 \\ -0.0475 & -0.7814 & 0.6223 \\ 0.9974 & -0.0712 & -0.0133 \end{bmatrix} = R_2(10),$$

which directly illustrates that the attitudes of two spacecraft are synchronized.



(a) Error functions Ψ_α, Ψ_β

(b) Relative error vector e_Q

(c) Angular velocity Ω_1, Ω_2 (rad/sec)

(d) Control inputs (Nm)

Fig. 2. Simulation results for attitude synchronization (first component:solid, second component:dashed, third component:dotted)

APPENDIX

A. Proof of Proposition 4

a) Lyapunov function: Define

$$\mathcal{V} = \frac{1}{2}\Omega_1 \cdot J_1\Omega_1 + \frac{1}{2}\Omega_2 \cdot J_2\Omega_2 + k_\alpha\Psi_\alpha + k_\beta\Psi_\beta.$$

From (11), (14), this is positive definite about the equilibrium $(Q, \Omega_1, \Omega_2) = (Q_d, 0, 0)$. From (13), (16), the time-derivative of $\mathcal{V}$ is given by

$$\begin{aligned}\dot{\mathcal{V}} &= \Omega_1 \cdot (-\Omega_1 \times J_1\Omega_1 + u_1) + \Omega_2 \cdot (-\Omega_2 \times J_2\Omega_2 + u_2) \\ &\quad + k_\alpha(\psi_{\alpha_1} \cdot \Omega_1 + \psi_{\alpha_2} \cdot \Omega_2) + k_\beta(\psi_{\beta_1} \cdot \Omega_1 + \psi_{\beta_2} \cdot \Omega_2).\end{aligned}$$

Substituting (13), (17), (18), and using (6), we obtain

$$\dot{\mathcal{V}} = -k_{\Omega_1}\|\Omega_1\|^2 - k_{\Omega_2}\|\Omega_2\|^2 \leq 0. \quad (24)$$

Therefore, the equilibrium $(Q_d, 0, 0)$ is stable, and according to LaSalle's theorem, the angular velocities asymptotically converges to zero, i.e. $\Omega_1, \Omega_2 \to 0$ as $t \to \infty$, and the attitude of each spacecraft asymptotically converges to the largest invariant set in the following set E:

$$\begin{aligned}E = \{(R_1, R_2) \in \mathsf{SO}(3)^2 \,|\, &k_\alpha\psi_{\alpha_1} + k_\beta\psi_{\beta_1} = 0, \\ &k_\alpha\psi_{\alpha_2} + k_\beta\psi_{\beta_2} = 0\}. \quad (25)\end{aligned}$$

b) Characteristics of the Invariant Set: Now we show the following two statements are equivalent, provided that $k_\alpha \neq k_\beta$ as stated at Proposition 4:

$$(k_\alpha\psi_{\alpha_1} + k_\beta\psi_{\beta_1} = 0) \quad (26)$$

$$\iff (\psi_{\alpha_1} = 0 \text{ and } \psi_{\beta_1} = 0). \quad (27)$$

First, suppose (27) is true, then (26) is trivial ($\Leftarrow$). Second, suppose (26) holds. We show (27) by contradiction: assume that (27) is false, i.e.

$$(\psi_{\alpha_1} = 0 \text{ and } \psi_{\beta_1} \neq 0), \text{ or} \quad (28)$$

$$(\psi_{\alpha_1} \neq 0 \text{ and } \psi_{\beta_1} = 0), \text{ or} \quad (29)$$

$$(\psi_{\alpha_1} \neq 0 \text{ and } \psi_{\beta_1} \neq 0). \quad (30)$$

It is clear that (28) and (29) imply (26) is not true.

So, from now on, we focus on (30), and we consider the following two cases of (30):

$$\text{Case 1:} \quad (\psi_{\beta_1} \cdot Q_d^T b_{21}) \neq 0 \text{ or } (\psi_{\beta_1} \cdot b_{12}) \neq 0, \quad (31)$$

$$\text{Case 2:} \quad (\psi_{\beta_1} \cdot Q_d^T b_{21}) = 0 \text{ and } (\psi_{\beta_1} \cdot b_{12}) = 0. \quad (32)$$

Each case is considered as follows.

Case 1: For Case 1, consider

$$\psi_{\alpha_1} \times \psi_{\beta_1} = (\psi_{\beta_1} \cdot Q_d^T b_{21})b_{12} - (\psi_{\beta_1} \cdot b_{12})Q_d^T b_{21}.$$

From (30), we have $\psi_{\alpha_1} = Q_d^T b_{21} \times b_{12} \neq 0$, which implies that two vectors $\{Q_d^T b_{21}, b_{12}\}$ are linearly independent. Together with (31), we have $\psi_{\alpha_1} \times \psi_{\beta_1} \neq 0$, which implies that two vectors $\{\psi_{\alpha_1}, \psi_{\beta_1}\}$ are linearly independent. This contradicts to (26) as the constants k_α, k_β are positive. In short, for Case 1 of (30), we have (26) is false.

Case 2: Now, consider Case 2. The definition of ψ_{β_1} is given by (17), and it is copied here as

$$\psi_{\beta_1} = \frac{1}{c}(Q_d^T(b_{21} \times b_{23})) \times (b_{12} \times b_{13}).$$

From (9), (10), the plane $\mathcal{P}_1$ is normal to $(b_{12} \times b_{13})$. Let the plane $\mathcal{P}_{d_2}$ be spanned by $Q_d^T b_{21}$ and $Q_d^T b_{23}$, such that it is normal to $(Q_d^T(b_{21} \times b_{23}))$. This implies that ψ_{β_1} is along the direction of the intersection between $\mathcal{P}_1$ and $\mathcal{P}_{d_2}$, and the magnitude of ψ_{β_1} is equal to $\|\psi_{\beta_1}\| = |\sin\theta_{12}|$, where θ_{12} is the angle between $\mathcal{P}_1$ and $\mathcal{P}_{d_2}$.

On the other hand, b_{12} lies in $\mathcal{P}_1$, and $Q_d^T b_{21}$ lies in $\mathcal{P}_{d_2}$ from (9), (10). Furthermore, the condition for Case 2, namely (32), implies that the direction of the intersection of $\mathcal{P}_1$ and $\mathcal{P}_{d_2}$, namely ψ_{β_1} is perpendicular to both of b_{12} and $Q_d^T b_{21}$. Then, the magnitude of $\psi_{\alpha_1} = Q_d^T b_{21} \times b_{12}$ is also equal to $\|\psi_{\alpha_1}\| = |\sin\theta_{12}|$.

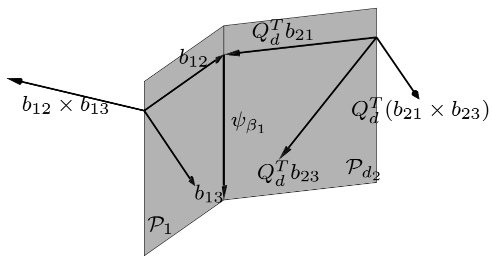
In short, for Case 2, the following four vectors are coplanar: b_{12}, $Q_d^T b_{21}$, $Q_d^T(b_{21} \times b_{23})$, $b_{12} \times b_{13}$. As a result, we have $\|\psi_{\alpha_1}\| = \|\psi_{\beta_1}\|$, which yields $\|k_\alpha\psi_{\alpha_1}\| \neq \|k_\beta\psi_{\beta_1}\|$ since $k_\alpha \neq k_\beta$. This contradicts with (26).

In summary, from (28), (29), and two cases of (30), if (27) is false, then (26) is false. Therefore, (26) implies (27) ($\Rightarrow$). This shows that (26) is equivalent to (27). Similarly,

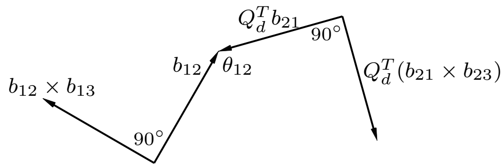
$$(k_\alpha\psi_{\alpha_2} + k_\beta\psi_{\beta_2} = 0) \iff (\psi_{\alpha_2} = 0 \text{ and } \psi_{\beta_2} = 0). \quad (33)$$

c) Asymptotic stability: From (26), (27), (33), the set E at (25) can be re-defined as (21). Consequently, there are four possible relative attitude in E. First, the desired relative attitude Q_d lies in E. But, there are three other (undesired) relative attitudes in E, which correspond to (a) rotation of Q_d about b_{12} by 180°, (b) rotation of Q_d about the direction of $b_{12} \times b_{13}$ by 180°, and (c) combination of two rotations at (a) and (b).

The four relative attitudes in E correspond to the critical points of $k_\alpha\Psi_\alpha + k_\beta\Psi_\beta$. At the desired relative attitude Q_d, we have $k_\alpha\Psi_\alpha + k_\beta\Psi_\beta = 0$, and at the other three undesired relative attitudes in E, $k_\alpha\Psi_\alpha + k_\beta\Psi_\beta$ has either one of the



(a) 3D view



(b) Top view (plane normal to ψ_{β_1})

Fig. 3. Illustration of one configuration for Case 2 to show $\|\psi_{\alpha_1}\| = \|\psi_{\beta_1}\|$: From (32), we have $\psi_{\beta_1} \perp b_{12}$, and $\psi_{\beta_1} \perp Q_d^T b_{21}$ (top view). Since $b_{12} \times b_{13}$ is normal to the plane $\mathcal{P}_1$ and the vector ψ_{β_1} lies on $\mathcal{P}_1$, we have $\psi_{\beta_1} \perp (b_{12} \times b_{13})$. Similarly, for $\mathcal{P}_{d_2}$, we also have $\psi_{\beta_1} \perp Q_d^T(b_{21} \times b_{23})$. In short, four vectors, namely b_{12}, $Q_d^T b_{21}$, $Q_d^T(b_{21} \times b_{23})$, $b_{12} \times b_{13}$, lies on the same plane perpendicular to ψ_{β_1} (top view). Therefore, $\|\psi_{\alpha_1}\| = \|Q_d^T b_{21} \times b_{12}\| = |\sin\theta_{12}|$, and $\|\psi_{\beta_1}\| = \frac{1}{c}\|(Q_d^T(b_{21} \times b_{23})) \times (b_{12} \times b_{13})\| = |\sin\theta_{12}|$, i.e. $\|\psi_{\alpha_1}\| = \|\psi_{\beta_1}\|$.

following values: $\{2k_\alpha,\, 2k_\beta,\, 2k_\alpha + 2k_\beta\}$. But, for the given estimate to the region of attraction, namely (22) and (23), we have $\mathcal{V}(0) < \min\{2k_\alpha,\, 2k_\beta\}$. Therefore, from (24),

$$k_\alpha \Psi_\alpha(t) + k_\beta \Psi_\beta(t) \le \mathcal{V}(t) \le \mathcal{V}(0) < \min\{2k_\alpha,\, 2k_\beta\}.$$

This guarantees that the three undesired relative attitudes are avoided. Any solution of the controlled system, if its initial condition satisfies (22), (23), we have $Q \to Q_d$ as $t \to \infty$. Therefore, the desired equilibrium is asymptotically stable.

d) Almost global asymptotic stability: Now we show that the three undesired relatives are unstable. It can be shown that the linearized system for an equilibrium of the proposed control system is unstable (and therefore, the equilibrium of the proposed nonlinear controlled system is unstable), if the hessian of $k_\alpha\Psi_\alpha + k_\beta\Psi_\beta$ is not positive semidefinite [13, Theorem 6.42]. The Hessian of each Ψ_α and Ψ_β can be written as a matrix form as follows:

$$[\delta^2\Psi_\alpha] = \begin{bmatrix} (Q_d^T b_{21})^\wedge \hat{b}_{12} & -\hat{b}_{12} Q_d^T \hat{b}_{21} \\ -\hat{b}_{21} Q_d \hat{b}_{12} & (Q_d b_{12})^\wedge \hat{b}_{21} \end{bmatrix},$$

$$[\delta^2\Psi_\beta] = \frac{1}{c}\begin{bmatrix} (Q_d^T b_{213})^\wedge \hat{b}_{123} & -\hat{b}_{123} Q_d^T \hat{b}_{213} \\ -\hat{b}_{213} Q_d \hat{b}_{123} & (Q_d b_{123})^\wedge \hat{b}_{213} \end{bmatrix},$$

where $b_{123} = b_{12} \times b_{13}$, $b_{213} = b_{21} \times b_{23}$.

At the first undesired relative attitude given by $b_{12} = +Q_d^T b_{21}$ and $\frac{b_{123}}{\|b_{123}\|} = -\frac{Q_d^T b_{213}}{\|b_{213}\|}$, the hessian of $k_\alpha\Psi_\alpha + k_\beta\Psi_\beta$ can be written as

$$k_\alpha \begin{bmatrix} \hat{b}_{12}^2 & -\hat{b}_{12}^2 Q_d^T \\ -Q_d \hat{b}_{12}^2 & ((Q_d b_{12})^\wedge)^2 \end{bmatrix} + \frac{k_\beta}{c}\begin{bmatrix} -\hat{b}_{123}^2 & \hat{b}_{123}^2 Q_d^T \\ Q_d \hat{b}_{123}^2 & -((Q_d b_{123})^\wedge)^2 \end{bmatrix}.$$

Consider a variation of two attitudes R_1, R_2 given by $\delta R_1 = R_1\hat{\eta}_1$, $\delta R_2 = R_2\hat{\eta}_2$, where $\eta_1, \eta_2 \in \mathbb{R}^3$ are chosen as $\eta_1 = b_{123}$ and $\eta_2 = 0$. Since $\hat{b}_{123} b_{123} = 0$, the corresponding variation of $k_\alpha\Psi_\alpha + k_\beta\Psi_\beta$, namely $\delta\Psi$ is given by

$$\begin{aligned} \delta\Psi &= k_\alpha b_{123}^T \hat{b}_{12}^2 b_{123} - \frac{k_\beta}{c} b_{123}^T \hat{b}_{123}^2 b_{123} \\ &= -k_\alpha \|b_{12} \times b_{123}\|^2 < 0, \end{aligned}$$

which states that the hessian of $k_\alpha\Psi_\alpha + k_\beta\Psi_\beta$ is not positive semidefinite. Therefore, the first undesired relative attitude is unstable. Similarly, we can show that two other remaining undesired relative attitudes are unstable as well.

Each of the three undesired relative attitude may have a stable manifold and an unstable manifold to itself. However, the union of the stable manifolds to the three undesired relative attitudes has a lower dimension than the tangent space of the configuration manifold, and therefore, its measure is zero. Any solution starting from outside of this zero-measure set asymptotically converges to the desired relative attitude. Then, the desired equilibrium $(Q, \Omega_1, \Omega_2) = (Q_d, 0, 0)$ is *almost* globally asymptotically stable [12].

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